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The heat equation with singular nonlinearity and singular initial data

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Abstract

We study the existence, uniqueness and regularity of positive solutions of the parabolic equation $u_t - \Delta u = a(x)u^q + b(x)u^p$ in a bounded domain and with Dirichlet's condition on the boundary. We consider here $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$ and $0 < q \leqslant 1 < p$. The initial data $u(0) = u_0$ is considered in the space $L^r(\Omega)$, $r \geqslant 1$. In the main result ($0 < q < 1$), we assume $a, b \geqslant 0$ a.e. in Ω and we assume that $u_0 \geqslant \gamma d_\Omega$ for some $\gamma > 0$. We find a unique solution in the space $C([0,T], L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$.

Keywords: Heat equation; Existence; Uniqueness; Concave–convex nonlinearity; Singular initial data

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$ and $T > 0$. We consider the following nonlinear heat equation

$$\begin{cases} u_t - \Delta u = a(x)u^q + b(x)u^p & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$, $\alpha, \beta \geqslant 1$, $0 < q \leqslant 1 < p$. We are interested in positive solutions.

Problems with the nonlinearity of (1.1) have been studied since the pioneering work of Ambrosetti, Brezis and Cerami [1]. These problems are important since they combine concavity and

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convexity effects, see also [7]. Problem (1.1) for $a = b = 1$ and $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ was studied by Cazenave, Dickstein and Escobedo [5]. They showed the existence of a unique positive solution $u \in L^\infty((0,T) \times \Omega)$ in a maximal time interval $[0, T_m)$. When $q = 1$, continuation of solutions after T_m and a priori estimates for problem (1.1) have been considered by Lópes and Quittner [9] and Quittner and Simondon [8].

We are interested in the existence, regularity and uniqueness of positive solutions for problem (1.1). In the main result, the initial data is assumed to satisfy $u_0 \in L^r(\Omega)$, $r \geqslant 1$, and $u_0 \geqslant \gamma d_\Omega$ where $\gamma > 0$ is a constant and

$$d_\Omega(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega. \tag{1.2}$$

When $a = 0$, $b = 1$, problem (1.1) has been considered by different authors (see [2,4, 6,10,12]), since the pioneering works of Weissler [13,14]. It is known that if $u_0 \in L^r(\Omega)$, $r > \frac{N}{2}(p-1)$ or $r = \frac{N}{2}(p-1)$ with $r > 1$, then there exists a unique solution u of (1.1) such that $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ with $u(0) = u_0$. Moreover, if $u_0 \geqslant 0$, then u is nonnegative. Here, we find analogous conditions for the existence and uniqueness of a solution of problem (1.1).

When $q = 1 < p$, one considers the problem

$$\begin{cases} u_t - \Delta u = a(x)u + b(x)|u|^{p-1}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.3}$$

In this case, the analysis is simple since the nonlinearity satisfies a Lipschitz condition. By a solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ of (1.3), we mean that

$$\begin{cases} u(t) = S(t-\epsilon)u(\epsilon) + \int_\epsilon^t S(t-\sigma)[au(\sigma) + b|u|^{p-1}u(\sigma)]\,d\sigma \quad \text{for } 0 < \epsilon \leqslant t \leqslant T, \\ u(t) \to u_0 \quad \text{in } L^r(\Omega) \text{ as } t \to 0, \end{cases}$$

where $(S(t))_{t \geqslant 0}$ is the linear heat semigroup on Ω with the Dirichlet condition on $\partial\Omega$. We have the following result.

Theorem 1.1. *Let $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$ with $1 < \alpha, \beta \leqslant \infty$. Assume that $u_0 \in L^r(\Omega)$, $1 \leqslant r < \infty$, $\alpha > \frac{N}{2}$ and $\frac{1}{\alpha} + \frac{1}{r} \leqslant 1 + \frac{2}{Np}$. If $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ with $r > 1$, then there exist $T > 0$ and a unique solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ of (1.3). Moreover, there exists a positive constant C such that*

$$t^{\frac{N}{2}(\frac{1}{r} - \frac{1}{s})} \|u(t)\|_{L^s} \leqslant C$$

for all $t \in (0,T]$ and $r \leqslant s \leqslant \infty$.

When $0 < q < 1$ the nonlinearity is not Lipschitz. In order to overcome the obstacle generated by the lack of the Lipschitz condition, we consider initial data in $u_0 \in L^r(\Omega)$ such that $u_0 \geqslant \gamma d_\Omega$ for some constant $\gamma > 0$. We also consider $a, b \geqslant 0$ a.e. in Ω. By a positive solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$ of (1.1), we mean that

$$\begin{cases} u(t) = S(t-\epsilon)u(\epsilon) + \int_\epsilon^t S(t-\sigma)[au^q(\sigma) + bu^p(\sigma)]\,d\sigma \quad \text{for } 0 < \epsilon \leqslant t \leqslant T, \\ u(t) \to u_0 \quad \text{in } L^r(\Omega) \text{ as } t \to 0. \end{cases} \tag{1.4}$$

Thus, we have:

Theorem 1.2. *Let $a \in L^{\alpha}(\Omega)$, $b \in L^{\beta}(\Omega)$ with $1 < \alpha, \beta \leqslant \infty$, $a, b \geqslant 0$ a.e. in Ω. Assume that $u_0 \in L^r(\Omega)$, $1 \leqslant r < \infty$ and there exists $\gamma > 0$ such that $u_0 \geqslant \gamma d_\Omega$ a.e. in Ω. Moreover, suppose $\alpha > \frac{N}{q+1}$ and $\frac{1}{\alpha} + \frac{q}{r} \leqslant q + \frac{1-q}{N} + \frac{2q}{Np}$. If $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$, then there exist $T = T(u_0) > 0$ and a positive solution $u \in C([0,T], L^r(\Omega)) \cap L^{\infty}_{\mathrm{loc}}((0,T), L^{\infty}(\Omega))$ of (1.1).*

Moreover, there exist $C, \gamma_1 > 0$ such that for $t \in (0, T]$

(i) *$u(t) \geqslant \gamma_1 d_\Omega$,*
(ii) *if $r \leqslant s \leqslant \infty$, then $t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}\|u(t)\|_{L^s} \leqslant C$,*
(iii) *if $N \geqslant 2$, then $t^{\frac{N}{2r}}\|u(t) - S(t)u_0\|_{W_0^{1,N}} \leqslant C$, and if $N = 1$ and $m \leqslant s \leqslant \infty$, then $t^{\frac{1}{2}+\frac{1}{2}(\frac{1}{r}-\frac{1}{s})}\|D_x[u(t) - S(t)u_0]\|_{L^s} \leqslant C$.*

This solution is unique in the class of functions

$$C\big([0,T], L^r(\Omega)\big) \cap L^{\infty}_{\mathrm{loc}}\big((0,T), L^{\infty}(\Omega)\big)$$

such that $u(t) \geqslant \gamma d_\Omega$ for t a.e. in $(0, T)$ for some $\gamma > 0$.

The space $W_0^{1,m}(\Omega)$, $m \geqslant 1$, denotes the closure of $C_0^1(\Omega)$ in the Sobolev space $W^{1,m}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,m}} = \|\nabla u\|_{L^m},$$

for all $u \in W_0^{1,m}(\Omega)$. In the proof of Theorem 1.2, we use a fixed point argument in a complete metric space $K(T)$ contained in $C((0,T), L^{\eta}(\Omega)) \cap C((0,T), W_0^{1,m}(\Omega))$ for well-chosen η, m and $T > 0$. In our estimates we will use the Hardy inequality, that is, there exists $C > 0$ such that

$$\left\| \frac{u}{d_\Omega} \right\|_{L^m} \leqslant C\|\nabla u\|_{L^m} \tag{1.5}$$

for all $u \in W_0^{1,m}(\Omega)$, $1 < m < \infty$. The function d_Ω is given by (1.2).

When a, b are positive constants, that is, $\alpha = \beta = \infty$, Theorem 1.2 is optimal. This follows from [14, Theorem 1] since the nonlinearity of (1.1) is larger than bu^p.

The plan of the paper is the following. In Section 2 we present some preliminary results and in Sections 3 and 4, we prove Theorems 1.2 and 1.1, respectively.

2. Preliminary results

We will frequently use the smoothing effect of the semigroup $(S(t))_{t \geqslant 0}$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If $1 \leqslant r, s \leqslant \infty$ and $u_0 \in L^r(\Omega)$, then $S(t)u_0 \in L^s(\Omega)$ and there exists a positive constant $C = C(|\Omega|)$ such that*

$$\|S(t)u_0\|_{L^s} \leqslant C t^{-\frac{N}{2}\max\{\frac{1}{r}-\frac{1}{s},0\}}\|u_0\|_{L^r} \quad \textit{for all } t > 0.$$

For the proof see [12, Lemma 4.4].

We also use the following lemma.

Lemma 2.2. *Given a compact set* $\mathcal{K} \subset L^r(\Omega)$ *and* $1 \leqslant r < s \leqslant \infty$*, there exists a function* $\gamma : (0,1] \to (0,\infty)$ *with* $\lim_{t\to 0} \gamma(t) = 0$ *such that* $t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}\|S(t)u_0\|_{L^s} \leqslant \gamma(t)$ *for all* $t \in (0,1)$ *and* $u_0 \in \mathcal{K}$.

For the proof see [4, Lemma 8].

Lemma 2.3. *Let* $\Omega \subset \mathbb{R}^N$ *be a* C^1 *bounded domain and* $f \in L^1((0,T), L^1(\Omega))$, $T > 0$. *Define for* $t \in (0,T)$,

$$w(t) = \int_0^t S(t-\sigma) f(\sigma)\, d\sigma.$$

If $w(t) \in L^m(\Omega)$ *for some* $1 < m < \infty$ *and* $\nabla S(t-\cdot) f(\cdot) \in L^1((0,t), L^m(\Omega))$, *then* $w(t) \in W_0^{1,m}(\Omega)$ *for every* $t \in (0,T)$.

Proof. Fix $t \in (0,T)$. Since $f \in L^1((0,T), L^1(\Omega))$, we have $S(t-\cdot) f \in L^1((0,t), L^1(\Omega))$. Thus, the function $w(t)$ is well defined. Moreover, by the regularity of Lemma 2.1, it follows that $S(t-\sigma) f(\sigma) \in W_0^{1,m}(\Omega)$ for σ a.e. in $(0,t)$.

On the other hand, we have that if $u \in W_0^{1,m}(\Omega)$, then

$$\left| \int_\Omega u \frac{\partial \varphi}{\partial x_i} \right| \leqslant \|\nabla u\|_{L^m} \|\varphi\|_{L^{m'}}, \quad i = 1, 2, \ldots, N, \tag{2.1}$$

for all $\varphi \in C_0^1(\mathbb{R}^N)$. Indeed, inequality (2.1) clearly holds for $u \in C_0^\infty(\Omega)$ and thus by density for $u \in W_0^{1,m}(\Omega)$. Using Fubini's theorem and inequality (2.1), we have

$$\left| \int_\Omega w(t) \frac{\partial \varphi}{\partial x_i} \right| \leqslant \|\varphi\|_{L^{m'}} \int_0^t \left\| \nabla S(t-\sigma) f(\sigma) \right\|_{L^m} d\sigma = C_t \|\varphi\|_{L^{m'}}$$

for all $\varphi \in C_0^1(\mathbb{R}^N)$. Since $w(t) \in L^m(\Omega)$ the desired result follows from [3, Proposition IX.18]. □

We will also use the following generalized Gronwall inequality [4].

Lemma 2.4. *Let* $T > 0$, $A \geqslant 0$, $\alpha \geqslant 0$, $0 \leqslant \beta, \gamma < 1$. *Consider* $\varphi \in L^\infty(0,T)$ *a nonnegative function such that*

$$\varphi(t) \leqslant A + t^\alpha \int_0^t (t-\sigma)^{-\beta} \sigma^{-\gamma} \varphi(\sigma)\, d\sigma \quad \text{a.e. in } (0,T).$$

If $1+\alpha > \beta+\gamma$, *then there exists a positive constant* $C = C(T,\alpha,\beta,\gamma) > 0$ *such that*

$$\varphi(t) \leqslant CA \quad \textit{a.e. in } (0,T).$$

For the proof of Theorems 1.1 and 1.2 we also need some more technical results.

Lemma 2.5. *Let* $0<q<1<p$ *and* $\alpha,\beta,s \geqslant 1$ *satisfying* $\frac{1}{\beta}+\frac{p}{s}<1$, $\frac{1}{\alpha}+\frac{q}{s}<q+\frac{1-q}{N}$, $\alpha>\frac{N}{q+1}$, $\frac{1}{\beta}+\frac{p-1}{s}<\frac{2}{N}$. *Let* $m(s)$ *given by*

$$\frac{1}{m(s)} = \begin{cases} \min\{\frac{1}{s}+\frac{1}{N}, 1-\frac{1}{N}\}, & N \geqslant 2, \\ 1-\frac{1}{\alpha}-\frac{q}{s}, & N=1. \end{cases} \tag{2.2}$$

Then,

(i) $\frac{1}{\alpha}+\frac{q}{s}+\frac{1-q}{m(s)} \leqslant 1$,
(ii) $\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m(s)}<\frac{1}{N}$,
(iii) $\frac{1}{m(s)}<\frac{1}{s}+\frac{1}{1-q}(\frac{2}{N}-\frac{1}{\alpha})$,
(iv) $\frac{p}{s}+\frac{1}{\beta}-\frac{1}{N}<\frac{1}{m(s)}$.

Proof. It follows by straightforward computations. $\square$

Remark 2.6. It is possible to find $\beta_0 \in [1,\beta]$ satisfying items (i)–(iv) of Lemma 2.5 and the condition $\frac{1}{m(s)} \leqslant \frac{1}{\beta_0}+\frac{p}{s}$. Indeed, if $\frac{1}{m(s)} > \frac{1}{\beta}+\frac{p}{s}$, then choosing $\beta_0 \in [1,\beta)$ such that $\frac{1}{\beta_0}+\frac{p}{s} = \frac{1}{m(s)} < 1$ we have $\frac{1}{\beta_0}+\frac{p}{s} \leqslant \frac{1}{s}+\frac{1}{N} < \frac{1}{s}+\frac{2}{N}$ for $N \geqslant 2$, and $\frac{1}{\beta_0}+\frac{p-1}{s} = 1-\frac{1}{\alpha}-\frac{1+q}{s} < 2$ for $N=1$.

Remark 2.7. For $N=1$, if $m \in (1,\infty)$ satisfies property (i) of Lemma 2.5, then it automatically satisfies properties (ii)–(iv).

Lemma 2.8. *Assume the hypothesis of Lemma* 2.5. *If* $m(s)>1$ *is given by* (2.2), $\tilde\beta = \frac{1}{2}+\frac{N}{2r}-\frac{N}{2m(s)}$ *and* $\tilde\alpha = \frac{N}{2}(\frac{1}{r}-\frac{1}{s})$, *then the expressions*

(i) $1+\tilde\alpha(1-q)-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s},0\}$,
(ii) $\frac{1}{2}+\tilde\beta-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{1}{m(s)},0\}-\tilde\alpha q$,
(iii) $1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m(s)},0\}+(1-q)(\tilde\alpha-\tilde\beta)$,
(iv) $\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m(s)},0\}+q(\tilde\beta-\tilde\alpha)$

are positive.

Proof. It is straightforward, using the fact $1 \geqslant \tilde\beta \geqslant \tilde\alpha$. $\square$

Lemma 2.9. *Assume that* $0 < q \leqslant 1 < p$, $\alpha, \beta, r \geqslant 1$ *with* $\alpha > \frac{N}{q+1}$ *and* $\frac{1}{\alpha} + \frac{q}{r} < q + \frac{1-q}{N} + \frac{2q}{Np}$. *If* $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ *or* $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ *and* $r > 1$, *then there exists* $\eta > r$ *such that*

(i) $\frac{1}{\alpha} + \frac{q}{\eta} < q + \frac{1-q}{N}$,
(ii) $\frac{1}{\beta} + \frac{p}{\eta} < 1$,
(iii) $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$,
(iv) $p\frac{N}{2}(\frac{1}{r} - \frac{1}{\eta}) < 1$.

Proof. Since $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$ or $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$ we have that $\frac{1}{\beta} + \frac{p}{r} < 1 + \frac{2}{N}$. Together the other hypothesis allow us to choose $\eta > r$ such that $\frac{1}{r} - \frac{2}{Np} < \frac{1}{\eta} < \frac{1}{p-1}(\frac{2}{N} - \frac{1}{\beta})$, $\frac{1}{\eta} < \frac{1}{p\beta'}$ and $\frac{1}{\eta} < 1 + \frac{1-q}{Nq} - \frac{1}{\alpha q}$. $\square$

The following result will be necessary to show the uniqueness of Theorem 1.2.

Proposition 2.10. *Assume that* $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$, $1 \leqslant \alpha, \beta, s \leqslant \infty$ *and* $0 < q < 1 < p$. *If* $u_0 \in L^s(\Omega)$, $\frac{1}{\beta} + \frac{p}{s} < 1$, $\alpha > \frac{N}{q+1}$, $\frac{1}{\alpha} + \frac{q}{s} < q + \frac{1-q}{N}$ *and* $\frac{1}{\beta} + \frac{p-1}{s} < \frac{2}{N}$, *then the problem*

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)\big[au^q(\sigma) + bu^p(\sigma)\big]\,d\sigma \tag{2.3}$$

has a unique solution in the class of functions

$$u \in L^\infty\big((0,T), L^s(\Omega)\big) \cap L^\infty_{\mathrm{loc}}\big((0,T), W_0^{1,m(s)}(\Omega)\big) \tag{2.4}$$

such that

$$\operatorname*{sup\,ess}_{t\in(0,T)} t^{\tilde\beta}\big\|u(t) - S(t)u_0\big\|_{W_0^{1,m(s)}} < \infty \tag{2.5}$$

and $u(t) \geqslant \gamma d_\Omega$ *for some* $\gamma > 0$ *and* $t \in (0,T)$. *The value* $m(s)$ *is defined by* (2.2) *and* $\tilde\beta = \frac{1}{2} + \frac{N}{2s} - \frac{N}{2m}$.

Proof. Let u and v be two solutions of Eq. (2.3) in the class (2.4), satisfying (2.5) and $u(t), v(t) \geqslant \gamma d_\Omega$ for $t \in (0,T)$. Then,

$$u(t) - v(t) = \underbrace{\int_0^t S(t-\sigma)a\big[u^q(\sigma) - v^q(\sigma)\big]\,d\sigma}_{W_1(t)} + \underbrace{\int_0^t S(t-\sigma)b\big[u^p(\sigma) - v^p(\sigma)\big]\,d\sigma}_{W_2(t)}. \tag{2.6}$$

Let $M = \operatorname{sup\,ess}_{t\in[0,T]}\{\|u(t)\|_{L^s}, \|v(t)\|_{L^s}\}$ and

$$\varphi(t) = \sup_{\sigma\in[0,t]} \big\|u(\sigma)-v(\sigma)\big\|_{L^s} + \operatorname*{sup\,ess}_{\sigma\in[0,t]} \sigma^{\tilde{\beta}} \big\|u(\sigma)-v(\sigma)\big\|_{W_0^{1,m}}.$$

Since $u(t), v(t) \geqslant \gamma d_\Omega$ for $t \in (0,T)$, then

$$\big|u^q - v^q\big| \leqslant q\gamma^{q-1}\frac{|u-v|}{d_\Omega^{1-q}} = C|u-v|^q\left(\frac{|u-v|}{d_\Omega}\right)^{1-q}. \tag{2.7}$$

By Lemma 2.5(i)–(iii), we have $\frac{1}{\alpha}+\frac{q}{s}+\frac{1-q}{m} \leqslant 1$, $\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m} < \frac{2}{N}$, $\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m} < \frac{1}{N}$. Thus, using Lemma 2.1 and Hardy's inequality (1.5)

$$\begin{aligned}\big\|W_1(t)\big\|_{L^s} &\leqslant C\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}\|u-v\|_{L^s}^q \big\|\nabla(u-v)\big\|_{L^m}^{1-q}\,d\sigma \\ &\leqslant C\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)\,d\sigma, \end{aligned} \tag{2.8}$$

$$\begin{aligned} t^{\tilde{\beta}}\big\|W_1(t)\big\|_{W_0^{1,m}} &\leqslant C\|a\|_{L^\alpha}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}\|u-v\|_{L^s}^q\|u-v\|_{W_0^{1,m}}^{1-q}\,d\sigma \\ &\leqslant Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)\,d\sigma. \end{aligned} \tag{2.9}$$

Similarly, since

$$\big|u^p - v^p\big| \leqslant C\big(|u|^{p-1}+|v|^{p-1}\big)|u-v|, \tag{2.10}$$

by Lemma 2.5(iv) and Remark 2.6, we have that $\frac{p}{s}+\frac{1}{\beta}-\frac{1}{N} < \frac{1}{m} \leqslant \frac{1}{\beta}+\frac{p}{s}$. Therefore, we conclude

$$\begin{aligned}\big\|W_2(t)\big\|_{L^s} &\leqslant M^{p-1}\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\|u-v\|_{L^s}\,d\sigma \\ &\leqslant C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\varphi(\sigma)\,d\sigma, \end{aligned} \tag{2.11}$$

$$
\begin{aligned}
t^{\tilde{\beta}}\big\|W_2(t)\big\|_{W_0^{1,m}} &\leqslant (M+1)^{p-1}\|b\|_{L^\beta}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m})}\|u-v\|_{L^s}\,d\sigma \\
&\leqslant Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m})}\varphi(\sigma)\,d\sigma. \qquad (2.12)
\end{aligned}
$$

From inequalities (2.8), (2.9), (2.11) and (2.12),

$$
\begin{aligned}
\varphi(t) &\leqslant C\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)\,d\sigma \\
&\quad + Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}}\sigma^{-\tilde{\beta}(1-q)}\varphi(\sigma)\,d\sigma \\
&\quad + C\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\varphi(\sigma)\,d\sigma + Ct^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m})}\varphi(\sigma)\,d\sigma.
\end{aligned}
$$

By Lemma 2.8 (for $r=s$) we have that $1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{s}+\frac{1-q}{m},0\}-\tilde{\beta}(1-q)$, $\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{s}-\frac{q}{m},0\}-\tilde{\beta}(1-q)$ are positive. Also, we have that $\frac{1}{2}+\tilde{\beta}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{s}-\frac{1}{m}) = 1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})>0$. By Lemma 2.4 we have that $\varphi(t)=0$, that is, $u(t)=v(t)$ for $t\in[0,T]$. □

Proposition 2.11. *Assume that $a\in L^\alpha(\Omega)$, $b\in L^\beta(\Omega)$ with $\alpha,\beta\geqslant 1$, $u_0\in L^s(\Omega)$ and $s\geqslant 1$. If $\frac{1}{\alpha}+\frac{1}{s}\leqslant 1$, $\frac{1}{\beta}+\frac{p}{s}\leqslant 1$, $\alpha>\frac{N}{2}$ and $\frac{1}{\beta}+\frac{p-1}{s}<\frac{2}{N}$, then the problem*

$$
u(t)=S(t)u_0+\int_0^t S(t-\sigma)\big[au(\sigma)+b|u|^{p-1}u(\sigma)\big]\,d\sigma \qquad (2.13)
$$

has a unique solution in $L^\infty((0,T),L^s(\Omega))$.

Proof. Let $u,v\in L^\infty((0,T),L^s(\Omega))$ solutions of problem (2.13). Since

$$
\big||u|^{p-1}u-|v|^{p-1}v\big|\leqslant C\big(|u|^{p-1}+|v|^{p-1}\big)|u-v|
$$

for some $C>0$, by Lemma 2.1 we have

$$
\begin{aligned}
\big\|u(t)-v(t)\big\|_{L^s} &\leqslant C\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2\alpha}}\big\|u(\sigma)-v(\sigma)\big\|_{L^s}\,d\sigma \\
&\quad + CM^{p-1}\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{s})}\big\|u(\sigma)-v(\sigma)\big\|_{L^s}\,d\sigma,
\end{aligned}
$$

where $M=\sup\operatorname{ess}_{t\in(0,T)}\{\|u(t)\|_{L^s},\|v(t)\|_{L^s}\}$. So, the result follows from Lemma 2.4. □

3. Proof of Theorem 1.2

To prove Theorem 1.2, we follow the standard way to study problems with singular initial data. We use a fixed point argument for the mapping $u \to \Phi(u)$, defined by

$$\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-\sigma)\big[au^q(\sigma) + bu^p(\sigma)\big]\,d\sigma \tag{3.1}$$

in a suitable complete metric space (see [4,13,14]).

Proof of the existence part of Theorem 1.2. We consider two situations.

Case 1. $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$. Let C_m be a positive constant such that

$$\big\|\nabla S(t)\phi\big\|_{L^m} \leqslant C_m t^{-1/2}\|\phi\|_{L^m} \tag{3.2}$$

for all $\phi \in L^m(\Omega)$ with $m \geqslant 1$. Let $C_0, C_1 > 0$ be constants such that

$$C_0\varphi_1 < d_\Omega < C_1\varphi_1, \tag{3.3}$$

where φ_1 is the first eigenvector associated to the first eigenvalue λ_1 of the operator $-\Delta$ in $H_0^1(\Omega)$.

Let η given by Lemma 2.9 and let $m = m(\eta)$ given by (2.2). Thus, the results of Lemmas 2.5, 2.8 (for $s = \eta$) and 2.9 hold. On the other hand, since Ω is bounded we have the usual inclusion of L^p spaces. By Remark 2.6, we can assume that

$$\frac{1}{m} \leqslant \frac{1}{\beta} + \frac{p}{\eta}. \tag{3.4}$$

Now, fix $M \geqslant \|u_0\|_{L^r}$ and let

$$E = C\big((0,T), L^\eta(\Omega)\big) \cap C\big((0,T), W_0^{1,m}(\Omega)\big),$$
$$K = \big\{u \in E\colon\ u(t) \geqslant \gamma_1 d_\Omega,\ t^{\tilde\alpha}\big\|u(t)\big\|_{L^\eta} \leqslant M+1,\ t^{\tilde\beta}\big\|\nabla\big(u(t) - S(t)u_0\big)\big\|_{L^m} \leqslant 1 \text{ for } t \in (0,T)\big\}$$

with $\tilde\alpha = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$, $\tilde\beta = -\frac{N}{2m} + \frac{1}{2} + \frac{N}{2r}$ and $\gamma_1 = \gamma_0 C_0 C_1^{-1} e^{-\lambda_1}$.

We equip K with the distance

$$d(u,v) = \max\Big\{\sup_{0<t<T} t^{\tilde\alpha}\|u-v\|_{L^\eta},\ \sup_{0<t<T} t^{\tilde\beta}\big\|\nabla(u-v)\big\|_{L^m}\Big\}.$$

The pair (K,d) is a nonempty complete metric space.

For $u \in K$, we set $\Phi(u)$ defined by (3.1). We will show that $\Phi(K) \subset K$ and that Φ is a strict contraction.

From Lemma 2.9, we have $\frac{1}{\alpha} + \frac{q}{\eta} \leqslant 1$, $\frac{1}{\beta} + \frac{1}{\eta} < 1$, $\frac{1}{\alpha} + \frac{q-1}{\eta} < \frac{1}{\alpha} < \frac{2}{N}$, $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$ and $\tilde\alpha p < 1$. Thus, by Lemma 2.1

$$
\begin{aligned}
t^{\tilde{\alpha}}\big\|\Phi(u)(t)\big\|_{L^\eta} &\leqslant \|u_0\|_{L^r} + Ct^{\tilde{\alpha}}\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}\big\|u(\sigma)\big\|_{L^\eta}^q \\
&\quad + Ct^{\tilde{\alpha}}\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\big\|u(\sigma)\big\|_{L^\eta}^p\,d\sigma \\
&\leqslant M + Ct^{\tilde{\alpha}}\|a\|_{L^\alpha}\Big(\sup_{t\in(0,T)} t^{\alpha}\big\|u(t)\big\|_{L^\eta}\Big)^q\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}\sigma^{-\tilde{\alpha}q}\,d\sigma \\
&\quad + Ct^{\tilde{\alpha}}\|b\|_{L^\beta}\Big(\sup_{t\in(0,T)} t^{\tilde{\alpha}}\big\|u(t)\big\|_{L^\eta}\Big)^p\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\sigma^{-\tilde{\alpha}p}\,d\sigma \\
&\leqslant M + C\|a\|_{L^\alpha}t^{1+\tilde{\alpha}(1-q)-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}(M+1)^q \\
&\quad + C\|b\|_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p.
\end{aligned}
\tag{3.5}
$$

From Lemma 2.5(ii) and (iv) (with $s=\eta$) and (3.4), we have that $\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m}<\frac{1}{N}$ and $\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{N}<\frac{1}{m}\leqslant\frac{1}{\beta}+\frac{p}{\eta}$. By (3.2) and Lemma 2.1, we conclude

$$
\begin{aligned}
&t^{\tilde{\beta}}\int_0^t \big\|\nabla\big[S(t-\sigma)\big(au^q+bu^p\big)\big]\big\|_{L^m}\,d\sigma \\
&\quad\leqslant t^{\tilde{\beta}}C\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}}\|u\|_{L^\eta}^q\,d\sigma \\
&\qquad + t^{\tilde{\beta}}CC\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}\|u\|_{L^\eta}^p\,d\sigma \\
&\quad\leqslant C\|a\|_{L^\alpha}t^{\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q + C\|b\|_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p \quad\text{and} \\
&t^{\tilde{\beta}}\left\|\int_0^t S(t-\sigma)\big[au^q+bu^p\big]\,d\sigma\right\|_{L^m} \\
&\quad\leqslant Ct^{\tilde{\beta}}\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}}\|u\|_{L^\eta}^q\,d\sigma \\
&\qquad + t^{\tilde{\beta}}C\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}\|u\|_{L^\eta}^p\,d\sigma \\
&\quad\leqslant C\|a\|_{L^\alpha}t^{1+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q + C\|b\|_{L^\beta}t^{\frac{3}{2}-\frac{N}{2}(\frac{1}{\beta}-\frac{p-1}{r})}(M+1)^p.
\end{aligned}
\tag{3.6}
$$

Thus, by Lemma 2.3, we have $\Phi(u)(t) - S(t)u_0 \in W_0^{1,m}(\Omega)$. Moreover, by (3.6)

$$\begin{aligned} & t^{\tilde{\beta}} \big\| \nabla\big[\Phi(u)(t) - S(t)u_0\big]\big\|_{L^m} \\ & \quad \leqslant C\|a\|_{L^\alpha} t^{\frac{1}{2}+\tilde{\beta}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m},0\}-\tilde{\alpha}q}(M+1)^q C\|b\|_{L^\beta} t^{1-\frac{N}{2\beta}-\tilde{\alpha}p}(M+1)^p. \end{aligned} \tag{3.7}$$

Proceeding as in the derivation of (3.5), we have for $0 < \tau < t < T$,

$$\begin{aligned} & \left\| \int_\tau^t S(t-\sigma)\big(au^q + bu^p\big)\, d\sigma \right\|_{L^\eta} \\ & \quad \leqslant \|a\|_{L^\alpha}(M+1)^q \int_\tau^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}} \sigma^{-\tilde{\alpha}q}\, d\sigma \\ & \qquad + \|b\|_{L^\beta}(M+1)^p \int_\tau^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})} \sigma^{-\tilde{\alpha}p}\, d\sigma \to 0, \quad \text{as } t \to \tau. \end{aligned} \tag{3.8}$$

Therefore, $\Phi(u) - S(\cdot)u_0 \in C((0,T], L^\eta(\Omega))$ and so $\Phi(u) \in C((0,T], L^\eta(\Omega))$. Similarly, we can show

$$\left\| \int_\tau^t S(t-\sigma)\big(au^q + bu^p\big)\, d\sigma \right\|_{W_0^{1,m}} \to 0, \quad \text{as } t \to \tau > 0, \tag{3.9}$$

and therefore $\Phi(u) \in C((0,T], W_0^{1,m}(\Omega))$.

By Lemma 2.3 we have $\Phi(u)(t) \in W_0^{1,m}(\Omega)$ for $t \in (0,T)$. Moreover, since $u, a, b \geqslant 0$, we have from (3.3) that $\Phi(u)(t) \geqslant S(t)u_0 \geqslant \gamma_1 d_\Omega$ for $T < 1$. From inequalities (3.5), (3.7) and Lemma 2.8(i), (ii), we have that $\Phi(K) \subset K$, for $T < 1$ sufficiently small.

To show that Φ is a strict contraction, let $u, v \in K$. From (3.1) we have

$$\Phi(u)(t) - \Phi(v)(t) = \underbrace{\int_0^t S(t-\sigma)a\big[u(\sigma)^q - v(\sigma)^q\big]\, d\sigma}_{W_1(t)} + \underbrace{\int_0^t S(t-\sigma)b\big[u(\sigma)^p - v(\sigma)^p\big]\, d\sigma}_{W_2(t)}.$$

Since $u(t), v(t) \geqslant \gamma_1 d_\Omega$ we have that (2.7) holds. Moreover, by Lemma 2.5(i)–(iii), we have $\frac{1}{\alpha} + \frac{q}{\eta} + \frac{1-q}{m} \leqslant 1$, $\frac{1}{\alpha} + \frac{q-1}{\eta} + \frac{1-q}{m} < \frac{2}{N}$ and $\frac{1}{\alpha} + \frac{q}{r} - \frac{q}{m} < \frac{1}{N}$. Thus, proceeding similarly as the derivation of (2.8) and (2.9)

$$t^{\tilde{\alpha}} \big\| W_1(t) \big\|_{L^\eta} \leqslant \gamma_1^{q-1} \|a\|_{L^\alpha} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}} \|u-v\|_{L^\eta}^q \big\| \nabla(u-v)\big\|_{L^m}^{1-q}\, d\sigma$$

$$\leqslant C\|a\|_{L^\alpha}\Big(\sup_{0<t<T} t^{\tilde{\alpha}}\big\|u(t)-v(t)\big\|_{L^\eta}\Big)^q\Big(\sup_{0<t<T} t^{\tilde{\beta}}\big\|\nabla\big(u(t)-v(t)\big)\big\|_{L^m}\Big)^{1-q}$$

$$\times t^{\tilde{\alpha}}\int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}}\sigma^{-\tilde{\alpha}q-\tilde{\beta}(1-q)}\,d\sigma$$

$$\leqslant C\|a\|_{L^\alpha}d(u,v)t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)\tilde{\alpha}-\tilde{\beta}(1-q)}, \tag{3.10}$$

$$t^{\tilde{\beta}}\big\|\nabla W_1(t)w\big\|_{L^m}\leqslant C\|a\|_{L^\alpha}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}}\|u-v\|_{L^\eta}^q\big\|\nabla(u-v)\big\|_{L^m}^{1-q}$$

$$\leqslant C\|a\|_{L^\alpha}d(u,v)t^{\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}+q(\tilde{\beta}-\tilde{\alpha})}. \tag{3.11}$$

On the other hand, since (2.10) holds, $\frac{1}{\beta}+\frac{p}{\eta}<1$ and $0\leqslant\frac{1}{\beta}+\frac{p-1}{\eta}<\frac{2}{N}$ (Lemma 2.9(ii) and (iii)). Proceeding as in the derivation of (2.11) and (2.12)

$$t^{\alpha}\big\|W_2(t)\big\|_{L^\eta}\leqslant C\|b\|_{L^\beta}t^{\tilde{\alpha}}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\big(\|u\|_{L^\eta}^{p-1}+\|v\|_{L^\eta}^{p-1}\big)\|u-v\|_{L^\eta}\,d\sigma$$

$$\leqslant C\|b\|_{L^\beta}(M+1)^{p-1}\sup_{0<t<T} t^{\tilde{\alpha}}\big\|u(t)-v(t)\big\|_{L^\eta}t^{\tilde{\alpha}}\int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}\sigma^{-\tilde{\alpha}p}$$

$$\leqslant C\|b\|_{L^\beta}d(u,v)t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}, \tag{3.12}$$

$$t^{\tilde{\beta}}\big\|\nabla W_2(t)\big\|_{L^m}\leqslant C\|b\|_{L^\beta}t^{\tilde{\beta}}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}\big(\|u\|_{L^\eta}^{p-1}+\|v\|_{L^\eta}^{p-1}\big)\|u-v\|_{L^\eta}$$

$$\leqslant C\|b\|_{L^\beta}(M+1)^{p-1}d(u,v)t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}. \tag{3.13}$$

Thus, we have that

$$t^{\tilde{\alpha}}\big\|\Phi(u)(t)-\Phi(v)(t)\big\|_{L^\eta}$$
$$\leqslant C\|a\|_{L^\alpha}d(u,v)t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)(\tilde{\alpha}-\tilde{\beta})}+C\|b\|_{L^\beta}d(u,v)t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})},$$
$$t^{\tilde{\beta}}\big\|\nabla\big[\Phi(u)(t)-\Phi(v)(t)\big]\big\|_{L^m}$$
$$\leqslant C\|a\|_{L^\alpha}d(u,v)t^{\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}+q(\tilde{\beta}-\tilde{\alpha})}+C\|b\|_{L^\beta}(M+1)^{p-1}d(u,v)t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}.$$

Therefore, by Lemma 2.8(iii) and (iv), shrinking $T<1$ if necessary we get that the map Φ is a strict contraction. Therefore, Φ has a fixed point in K.

We now show that $u\in C([0,T],L^r(\Omega))$. Indeed, we have $u\in K$, so that in particular $u\in C((0,T],L^\eta(\Omega))\subset C((0,T],L^r(\Omega))$, since $\eta>r$. Therefore, it remains to show that

$u(t) - S(t)u_0 \to 0$ in $L^r(\Omega)$ as $t \to 0$. To this end, first note $\alpha > \frac{N}{q+1}$, $\frac{1}{\alpha} + \frac{q}{\eta} - \frac{1}{r} < \frac{2}{N}$ and by Lemma 2.9(iii), $\frac{1}{\beta} + \frac{p}{\eta} - \frac{1}{r} < \frac{2}{N}$. By Lemma 2.1

$$\begin{aligned}\left\|u(t) - S(t)u_0\right\|_{L^r} &\leqslant C\|a\|_{L^\alpha} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{r},0\}} \left\|u(\sigma)\right\|_{L^\eta}^q \, d\sigma \\ &\quad + C\|b\|_{L^\beta} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{r},0\}} \left\|u(\sigma)\right\|_{L^\eta}^p \, d\sigma \\ &\leqslant C\|a\|_{L^\alpha}(M+1)^q t^{1-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}q} \\ &\quad + C\|b\|_{L^\beta}(M+1)^p t^{1-\frac{N}{2}\max\{\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{r},0\}-\tilde{\alpha}p} \to 0, \quad \text{as } t \to 0. \end{aligned} \tag{3.14}$$

Case 2. $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ with $r > 1$. The argument is similar to the previous case with some minor technical differences. We only show the existence of a solution. The regularity and uniqueness part follow as in the previous case.

Let η given by Lemma 2.9, $m = m(\eta)$ given by (2.2) and

$$E = \left\{u \in C\big((0,T), L^\eta(\Omega)\big)\colon \lim_{t\to 0} t^{\tilde{\alpha}} u(t) = 0\right\} \cap C\big((0,T), W_0^{1,m}(\Omega)\big),$$

where $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$. Given $\delta > 0$ to be chosen later, let

$$K = \left\{u \in E\colon u(t) \geqslant \gamma_1 d_\Omega,\ t^{\tilde{\alpha}}\left\|u(t)\right\|_{L^\eta} \leqslant \delta,\ t^{\tilde{\beta}}\left\|\nabla\left[u(t) - S(t)u_0\right]\right\|_{L^m} \leqslant 1\right\},$$

γ_1 is defined as the previous case. The value $\tilde{\beta}$ satisfies $\tilde{\beta} + \frac{N}{2m} = \frac{1}{2} + \frac{N}{2r}$. We equip K with the distance

$$d(u,v) = \max\left\{\sup_{0<t<T} t^{\tilde{\alpha}}\left\|u(t) - v(t)\right\|_{L^\eta},\ \sup_{0<t<T} t^{\tilde{\beta}}\left\|\nabla\left[u(t) - v(t)\right]\right\|_{L^m}\right\}.$$

The pair (K,d) is a nonempty complete metric space. For $u \in K$ consider the application defined by (3.1). As the previous case, we have $\Phi(u)(t) \geqslant \gamma_1 d_\Omega$ and $\Phi(u)(t) - S(t)u_0 \in W_0^{1,m}(\Omega)$.

Proceeding as in the derivation of (3.5) and (3.7),

$$\begin{aligned} t^{\tilde{\alpha}}\left\|\Phi(u)(t)\right\|_{L^\eta} &\leqslant t^{\tilde{\alpha}}\left\|S(t)u_0\right\|_{L^\eta} + \|a\|_{L^\alpha} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}} \|u\|_{L^\eta}^q \\ &\quad + \|b\|_{L^\beta} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})} \|u\|_{L^\eta}^p \\ &\leqslant t^{\tilde{\alpha}}\left\|S(t)u_0\right\|_{L^\eta} + C\|a\|_{L^\alpha}\delta^q t^{1+\tilde{\alpha}(1-q)-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}} + C_1\|b\|_{L^\beta}\delta^p, \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
t^{\tilde\beta}\big\|\nabla\big[\Phi(u)(t)-S(t)u_0\big]\big\|_{L^m} &\leqslant t^{\tilde\beta}C_m\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac12-\frac N2\max\{\frac1\alpha+\frac q\eta-\frac1m,0\}}\|u\|_{L^\eta}^q \\
&\quad + \|b\|_{L^\beta}C_m t^{\tilde\beta}\int_0^t (t-\sigma)^{-\frac12-\frac N2(\frac1\beta+\frac p\eta-\frac1m)}\|u\|_{L^\eta}^p \\
&\leqslant C\|a\|_{L^\alpha}t^{\frac12+\tilde\beta-\frac N2\max\{\frac1\alpha+\frac q\eta-\frac1m,0\}-\tilde\alpha q}\delta^q + C_2\|b\|_{L^\beta}\delta^p. \qquad (3.16)
\end{aligned}
$$

Moreover, proceeding as in the derivations of inequalities (3.10)–(3.13) we have for $u, v \in K$

$$
\begin{aligned}
t^{\tilde\alpha}\big\|\Phi(u)(t)-\Phi(v)(t)\big\|_{L^\eta} &\leqslant C\|a\|_{L^\alpha}d(u,v)t^{1-\frac N2\max\{\frac1\alpha+\frac{q-1}{\eta}+\frac{1-q}{m},0\}+(1-q)(\tilde\alpha-\tilde\beta)} \\
&\quad + C_3\|b\|_{L^\beta}\delta^{p-1}d(u,v), \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
t^{\tilde\beta}\big\|\nabla\big(\Phi(u)(t)-\Phi(v)(t)\big)\big\|_{L^m} &\leqslant C\|a\|_{L^\alpha}d(u,v)t^{\frac12-\frac N2\max\{\frac1\alpha+\frac q\eta-\frac qm,0\}+q(\tilde\beta-\tilde\alpha)} \\
&\quad + C_4\|b\|_{L^\beta}\delta^{p-1}d(u,v). \qquad (3.18)
\end{aligned}
$$

Now, fix $\delta \in (0,1)$ such that $C\|b\|_{L^\beta}\delta^{p-1} < \frac\delta4$ where $C = \max\{C_1,\dots,C_4\}$. By Lemma 2.2 there exists $T > 0$ such that $t^{\tilde\alpha}\|S(t)u_0\|_{L^\eta} \leqslant \frac\delta4$. Thus, from (3.15) and (3.16) and Lemma 2.8 we have $t^{\tilde\alpha}\|\Phi(u)(t)\|_{L^\eta} \leqslant \delta$ and $t^{\tilde\beta}\|\nabla[\Phi(u)(t)-S(t)u_0]\|_{L^m} \leqslant 1$, for $T > 0$ small enough. Therefore, we have $\Phi(K) \subset K$. Moreover, from inequalities (3.17) and (3.18), shrinking T if necessary we have $d(\Phi(u),\Phi(v)) \leqslant \frac12 d(u,v)$, that is, the map Φ is a strict contraction. Therefore, it has a fixed point in K.

We use the same argument as the previous case for show that $u \in C((0,T], L^r(\Omega))$. It remains to show the continuity at the point $t = 0$. Proceeding as in the derivation of inequality (3.14), we have

$$
\begin{aligned}
\big\|u(t)-S(t)u_0\big\|_{L^r} &\leqslant \|a\|_{L^\alpha}(M+1)^q t^{1-\frac N2\max\{\frac1\alpha+\frac q\eta-\frac1r,0\}-\tilde\alpha q} \\
&\quad + C\|b\|_{L^\beta}\Big(\sup_{0<\sigma<t}\sigma^{\tilde\alpha}\big\|u(\sigma)\big\|_{L^\eta}\Big)^p t^{1-\frac N2\max\{\frac1\beta+\frac p\eta-\frac1r,0\}-\tilde\alpha p} \\
&\to 0, \quad \text{as } t \to 0,
\end{aligned}
$$

since $u \in E$. $\square$

Remark 3.1. In Case 1, the choice of T depends of $\|u_0\|_{L^r}$. In Case 2 the choice of T depends of the compact $\mathcal{K} \subset L^r(\Omega)$ that contains u_0.

For $u_0 \in L^\infty(\Omega)$ we have the following result.

Proposition 3.2. *Assume that $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$, $a, b \geqslant 0$ a.e. in Ω, $\alpha > \frac{N}{q+1}$, $\beta > \frac N2$ with $\alpha, \beta \geqslant 1$, $0 < q < 1 < p$. If $u_0 \in L^\infty(\Omega)$ and $u_0 \geqslant \gamma d_\Omega$ for some $\gamma > 0$, then there exist $T > 0$ and a function*

$$
u \in L^\infty\big((0,T), L^\infty(\Omega)\big) \cap L^\infty_{\mathrm{loc}}\big((0,T), W_0^{1,m(\infty)}(\Omega)\big) \qquad (3.19)
$$

satisfying Eq. (2.3). *This solution is unique in the class of functions* (3.19) *such that*

$$\sup_{t\in(0,T)} \operatorname{ess} t^{\tilde{\beta}} \left\| u(t) - S(t)u_0 \right\|_{W_0^{1,m(\infty)}} < \infty$$

and $u(t) \geqslant \gamma_1 d_\Omega$ *a.e. in* $(0,T)\times\Omega$ *for some* $\gamma_1 > 0$. *The value m is defined by* (2.2).

Proof. The existence can be shown adapting the arguments of the previous proof. The uniqueness follows from Proposition 2.10. □

Proof of the regularity of Theorem 1.2. We use a bootstrap argument, as in [11]. The existence proof ensures that for all $t \in (0,T]$

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta})} \left\| u(t) \right\|_{L^\eta} \leqslant C, \tag{3.20}$$

with $C = M+1$ in Case 1 and $C = \delta$ in Case 2. We will show that (3.20) continues still holds for some $\eta' > \eta$.

Let u be the solution obtained above. For $t \in (0,T]$

$$u(t) = S(t/2)u(t/2) + \int_{t/2}^{t} S(t-\sigma)\left[au^q(\sigma) + bu^p(\sigma)\right] d\sigma. \tag{3.21}$$

By the proof of Theorem 1.2, we have $\frac{1}{\alpha} + \frac{q}{\eta} - \frac{2}{N} < \frac{1}{\eta}$ and $\frac{1}{\beta} + \frac{p}{\eta} - \frac{2}{N} < \frac{1}{\eta}$. Then there exists $\eta' > \eta$ such that

$$\frac{1}{\alpha} + \frac{q}{\eta} - \frac{2}{N} < \frac{1}{\eta'} \leqslant \frac{1}{\alpha} + \frac{q}{\eta} \quad \text{and} \quad \frac{1}{\beta} + \frac{p}{\eta} - \frac{2}{N} < \frac{1}{\eta'} \leqslant \frac{1}{\beta} + \frac{p}{\eta}.$$

Since $\frac{1}{\beta} + \frac{p}{\eta} \leqslant 1$ and $\frac{1}{\alpha} + \frac{q}{\eta} \leqslant 1$, we have from (3.21), (3.20)

$$\begin{aligned}
\left\| u(t) \right\|_{L^{\eta'}} &\leqslant (t/2)^{-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{\eta'})} \left\| u(t/2) \right\|_{L^\eta} + \|a\|_{L^\alpha} \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})} \|u\|_{L^\eta}^q \, d\sigma \\
&\quad + \|b\|_{L^\beta} \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})} \|u\|_{L^\eta}^p \, d\sigma \\
&\leqslant (t/2)^{-\frac{N}{2}(\frac{1}{\eta}-\frac{1}{\eta'})} \left\| u(t/2) \right\|_{L^\eta} + C^q \|a\|_{L^\alpha} \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})} \sigma^{-\frac{Nq}{2}(\frac{1}{r}-\frac{1}{\eta})} \, d\sigma \\
&\quad + C^p \|b\|_{L^\beta} \int_{t/2}^{t} (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})} \sigma^{-\frac{Np}{2}(\frac{1}{r}-\frac{1}{\eta})} \, d\sigma.
\end{aligned}$$

Since

$$\int_{1/2}^{1} (1-\sigma)^{-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Nq}{2}(\frac{1}{r}-\frac{1}{\eta})}\,d\sigma<\infty, \qquad \int_{1/2}^{1} (1-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{\eta'})}\sigma^{-\frac{Np}{2}(\frac{1}{r}-\frac{1}{\eta})}\,d\sigma<\infty,$$

we have

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}\big\|u(t)\big\|_{L^{\eta'}}\leqslant Ct^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}+C\|a\|_{L^\alpha}t^{1-\frac{N}{2\alpha}+\frac{N(1-q)}{2r}}+C\|b\|_{L^\beta}t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}=C'.$$

Therefore, inequality (3.20) holds for $\eta'>\eta$. Thus, one can bootstrap in a finite number steps to obtain a constant $C>0$ such that $t^{\frac{N}{2r}}\|u(t)\|_{L^\infty}\leqslant C$. Since $\|u(t)\|_{L^r}\leqslant M+1$, using interpolation we conclude that there exists a constant $C>0$, depending of a, b, M, T such that

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}\big\|u(t)\big\|_{L^s}\leqslant C \tag{3.22}$$

for $r\leqslant s\leqslant\infty$ and $t\in(0,T]$.

Similarly, since $u\in K$, K defined in the proof of Theorem 1.2, we have

$$t^{\tilde\beta}\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^{m(s)}}\leqslant 1 \tag{3.23}$$

for all $t\in(0,T]$ with $\tilde\beta=\frac{N}{2}(\frac{1}{r}-\frac{1}{m(s)}+\frac{1}{N})$, where $m(s)$ is defined by (2.2) and $s=\eta$. We will show that (3.23) holds for some $s=\eta'>\eta$.

First, we consider the case $N>2$. From (3.21) we have

$$u(t)-S(t)u_0=S(t/2)\big[u(t/2)-S(t/2)u_0\big]+\int_{t/2}^{t}S(t-\sigma)\big[au^q(\sigma)+bu^p(\sigma)\big]\,d\sigma. \tag{3.24}$$

By Lemma 2.5(ii) and (iv), it is possible to choose $\eta'>\eta$ such that $\frac{1}{\eta'}+\frac{2}{N}\leqslant 1$ and $0\leqslant\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{m(\eta')}<\frac{1}{N}$, $0\leqslant\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m(\eta')}<\frac{1}{N}$. Since $m(\eta')>m(\eta)$ we have from (3.24) that

$$\begin{aligned}\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^{m(\eta')}}&\leqslant(t/2)^{-\frac{N}{2}(\frac{1}{m(\eta)}-\frac{1}{m(\eta')})}\big\|\nabla\big[u(t/2)-v(t/2)\big]\big\|_{L^{m(\eta)}}\\&\quad+C\|a\|_{L^\alpha}\int_{t/2}^{t}(t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\alpha}+\frac{q}{n}-\frac{1}{m(\eta')})}\big\|u(\sigma)\big\|_{L^\eta}^q\\&\quad+C\|b\|_{L^\beta}\int_{t/2}^{t}(t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m(\eta')})}\big\|u(\sigma)\big\|_{L^\eta}^p\,d\sigma.\end{aligned}$$

So by (3.22), we conclude that

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{\eta'})}\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^{m(\eta')}} \leqslant C + C\|a\|_{L^\alpha} t^{1-\frac{N}{2\alpha}+\frac{N(1-q)}{2r}} + C\|b\|_{L^\beta} t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}$$
$$\leqslant C'(T)$$

for $t\in(0,T]$. Therefore, we have that (3.23) holds for η'. By the bootstrap argument already used we conclude that there exists a constant $C>0$ such that

$$t^{\frac{N}{2r}}\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^N}\leqslant C.$$

For the case $N=2$, it is sufficient to replace the value $N=2$ in expression (3.23).

In the case $N=1$ we use the following argument. From Lemma 2.9(i) and (ii), we have $\frac{1}{\alpha}+\frac{q}{\eta}<1$ and $\frac{1}{\beta}+\frac{p}{\eta}<1$. Let $s>m(\eta)$ such that $\frac{1}{s}<\frac{1}{\alpha}+\frac{q}{\eta}$ and $\frac{1}{s}<\frac{1}{\beta}+\frac{p}{\eta}$. Then by Lemma 2.1, inequalities (3.23) and (3.24)

$$\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^s}\leqslant t^{-\frac{N}{2}(\frac{1}{m(\eta)}-\frac{1}{s})}\big\|\nabla\big[u(t/2)-S(t/2)u_0\big]\big\|_{L^{m(\eta)}}$$
$$+C\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\alpha}+\frac{q}{\eta}-\frac{1}{s})}\big\|u(\sigma)\big\|_{L^\eta}^q$$
$$+\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{s})}\big\|u(\sigma)\big\|_{L^\eta}^p\,d\sigma.$$

Thus

$$t^{\frac{1}{2}(1+\frac{1}{r}-\frac{1}{s})}\big\|\nabla\big[u(t)-S(t)u_0\big]\big\|_{L^s}\leqslant C+Ct^{1-\frac{N}{2}(\frac{1}{\alpha}+\frac{q-1}{r})}+Ct^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}\leqslant C(T). \qquad \square$$

Proof of the uniqueness of Theorem 1.2. Assume that $v\in C([0,T],L^r(\Omega))\cap L^\infty_{\mathrm{loc}}((0,T),L^\infty(\Omega))$ is a solution of (1.1) in the sense (1.4) and that $v(t)\geqslant\gamma d_\Omega$ for some $\gamma>0$.

We first show that there exists $T'>0$ such that $v(t)=u(t)$ for all $t\in[0,T']$. Set $W=v([0,T])$ and $M=\sup_{t\in[0,T]}\|v(t)\|_{L^r}$. Since that $W\subset L^r(\Omega)$ is a compact, by Remark 3.1 and the proof of existence of Theorem 1.2, there exists a uniform $T_1>0$ and for every $\tau\in(0,T)$ there exists a solution $v_\tau\in C([0,T_1],L^r(\Omega))$ of (2.3) such that

$$v_\tau\in C\big((0,T_1],L^\eta(\Omega)\big)\cap C\big((0,T_1],W_0^{1,m(\eta)}(\Omega)\big), \tag{3.25}$$

with $v_\tau(0)=v(\tau)$ and such that $v_\tau\in K(T_1)$.

On the other hand, from (1.4)

$$v(t+\tau)=S(t)v(\tau)+\int_0^t S(t-\sigma)\big[av^q(\sigma+\tau)+bv^p(\sigma+\tau)\big]\,d\sigma \tag{3.26}$$

for $\tau\in(0,T)$ and $0<t<T-\tau$.

For each $\tau \in (0,T)$, let $M_\tau = \sup_{t\in[\tau,T]} \|u(t)\|_{L^\eta}$. From (3.3), we have $\gamma_1 = \gamma C_0 C_1^{-1} \times e^{-\lambda_1} < \gamma$. Then, $v(t+\tau) \geqslant \gamma_1 d_\Omega$ for $t \in (0, T-\tau)$. Proceeding as in the derivation of (3.5) we have

$$t^{\tilde\alpha}\big\|v(t+\tau)\big\|_{L^\eta} \leqslant t^{\tilde\alpha}\big\|S(t)v(\tau)\big\|_{L^r} + Ct^{1+\tilde\alpha-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q-1}{\eta},0\}}M_\tau^q + Ct^{1+\tilde\alpha-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{\eta})}M_\tau^p.$$

Proceeding as in the derivation of (3.7), we also have

$$t^{\tilde\beta}\big\|v(t+\tau) - S(t)v(\tau)\big\|_{W_0^{1,m}} \leqslant Ct^{\frac{1}{2}+\tilde\beta+-\frac{N}{2}\max\{\frac{1}{\alpha}+\frac{q}{\eta}-\frac{q}{m},0\}}M_\tau^q + Ct^{\frac{1}{2}+\tilde\beta-\frac{N}{2}(\frac{1}{\beta}+\frac{p}{\eta}-\frac{1}{m})}M_\tau^p.$$

Therefore, by Lemma 2.2, there exists $T_\tau > 0$ such that $v(\cdot+\tau) \in K(T_\tau)$. By the uniqueness in $K(T'_\tau)$ with $T'_\tau = \min\{T_1, T_\tau\}$, we conclude that $v_\tau(t) = v(t+\tau)$ for all $t \in [0, \min\{T'_\tau, T-\tau\}]$. By Proposition 2.10, we have that the uniqueness holds in the class (3.25). Therefore, $v_\tau(t) = v(t+\tau)$ for all $t \in [0, \min\{T_1, T-\tau\}]$. Thus, since $v_\tau \in K(T_1)$, we have

$$t^{\tilde\alpha}\big\|v(t+\tau)\big\|_{L^\eta} \leqslant M+1, \qquad t^{\tilde\beta}\big\|v(t+\tau) - S(t)v(\tau)\big\|_{W_0^{1,m}} \leqslant 1$$

for $t \in (0, \min\{T_1, T-\tau\})$. Arguing as (3.8) and (3.9) we obtain the continuity of v that allows us to consider $\tau \to 0$. So, we deduce that $t^{\tilde\alpha}\|v(t)\|_{L^\eta} \leqslant M+1$ and $t^{\tilde\beta}\|v(t) - S(t)v_0\|_{W_0^{1,m}} \leqslant 1$ for all $t \in (0, \min\{T, T_1\})$, that is, $v \in K(\min\{T, T_1\})$ and v is the solution obtained by the fixed point argument. Thus, $v(t) = u(t)$ for all $t \in [0, T']$ with $T' = \min\{T, T_1\}$.

From (3.26) for $\tau = T'$ we have

$$\begin{aligned}\big\|v(t+T') - S(t)u(T')\big\|_{W_0^{1,m(\infty)}} &\leqslant C\|a\|_{L^\alpha}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\alpha}-\frac{1}{m(\infty)},0\}}\big\|v(\cdot+T')\big\|_{L^\infty}^q\,d\sigma \\ &\quad + C\|b\|_{L^\beta}\int_0^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}\max\{\frac{1}{\beta}-\frac{1}{m(\infty)},0\}}\big\|v(\cdot+T')\big\|_{L^\infty}^p\,d\sigma \\ &\leqslant C(T,T').\end{aligned}$$

By the uniqueness of Proposition 2.10, for $s = \infty$, we have that v is the unique solution after T' and therefore in $[0, T]$. □

4. Proof of Theorem 1.1

Proof of the existence of Theorem 1.1. We use the same argument that was used to show Theorem 1.2. We consider two cases.

Case 1. $\frac{1}{\beta} + \frac{p-1}{r} < \frac{2}{N}$. Fix $M \geqslant \|u_0\|_{L^r}$ and let $E = L^\infty((0,T), L^\eta(\Omega))$, where η is given by Lemma 2.9 with $q = 1$, $K = \{u \in E\colon t^{\tilde\alpha}\|u(t)\|_{L^\eta} \leqslant M+1\}$ and $\tilde\alpha = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$. We equip K with the distance $d(u,v) = \sup_{0<t<T} t^{\tilde{\alpha}} \|u(t) - v(t)\|_{L^\eta}$. In this way, the pair (K, d) is a nonempty complete metric space. Given $u \in K$, we set

$$\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-\sigma)\big[au(\sigma) + b\big|u(\sigma)\big|^{p-1}u(\sigma)\big]\,d\sigma.$$

Since $\frac{1}{\alpha} + \frac{1}{\eta} < 1$, $\alpha > \frac{N}{2}$, $\frac{1}{\beta} + \frac{p}{\eta} < 1$ and $\frac{1}{\beta} + \frac{p-1}{\eta} < \frac{2}{N}$, we have for $u \in K$

$$\begin{aligned} t^{\tilde{\alpha}} \big\|\Phi(u)(t)\big\|_{L^\eta} &\leqslant \|u_0\|_{L^r} + \|a\|_{L^\alpha} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2\alpha}} \|u\|_{L^\eta} \\ &\quad + \|b\|_{L^\beta} t^{\tilde{\alpha}} \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta} + \frac{p-1}{\eta})} \|u\|_{L^\eta}^p \, d\sigma \\ &\leqslant \|u_0\|_{L^r} + C\|a\|_{L^\alpha}(M+1)t^{1-\frac{N}{2\alpha}} + C\|b\|_{L^\beta} t^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})}(M+1)^p. \end{aligned}$$

Similarly, one can show that for $u, v \in K$

$$\begin{aligned} t^{\tilde{\alpha}} \big\|\Phi(u)(t) - \Phi(v)(t)\big\|_{L^\eta} &\leqslant C\|a\|_{L^\alpha} t^{1-\frac{N}{2\alpha}} \sup_{t\in(0,T)} t^{\tilde{\alpha}} \big\|u(t) - v(t)\big\|_{L^\eta} \\ &\quad + Ct^{1-\frac{N}{2}(\frac{1}{\beta}+\frac{p-1}{r})} \|b\|_{L^\beta}(M+1)^{p-1} \sup_{t\in(0,T)} t^{\tilde{\alpha}} \big\|u(t) - v(t)\big\|_{L^\eta}. \end{aligned}$$

It follows from the above estimates that if $T > 0$ is small enough then $\Phi(K) \subset K$ and Φ is a strict contraction. Thus, the map Φ has a fixed point in K.

The proof that $u \in C([0,T], L^r(\Omega))$ is completely analogous to the proof of Theorem 1.2.

Case 2. $\frac{1}{\beta} + \frac{p-1}{r} = \frac{2}{N}$ and $r > 1$. We proceed as the previous case with η given by Lemma 2.5 and using a contraction mapping principle in the space

$$K = \big\{u \in E\colon\ t^{\tilde{\alpha}} \big\|u(t)\big\|_{L^\eta} \leqslant \delta \text{ for } t \in (0,T)\big\},$$

where $\tilde{\alpha} = \frac{N}{2}(\frac{1}{r} - \frac{1}{\eta})$ and $E = \{u \in L^\infty((0,T), L^\eta(\Omega))\colon \lim_{t\to 0} t^{\tilde{\alpha}} u(t) = 0\}$. □

Using Proposition 2.11 and a similar argument as in the proof of Case 1 above, we have:

Proposition 4.1. *Assume that $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$ with $\alpha, \beta > \frac{N}{2}$, $\alpha, \beta \geqslant 1$ and $q = 1$. If $u_0 \in L^\infty(\Omega)$ then there exist a unique function $u \in L^\infty((0,T), L^\infty(\Omega))$ satisfying* (2.13).

Proof of regularity and uniqueness of Theorem 1.1. One proceeds as in the regularity part and uniqueness part in the proof of Theorem 1.2, using the Proposition 4.1 instead of Proposition 3.2. □

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